Shape Optimization with an Unfitted Finite Element Method

Jeremy T. Shahan
Advisor: Shawn W. Walker

Louisiana State University
Department of Mathematics and
Center for Computation and Technology (CCT)

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Model Problem

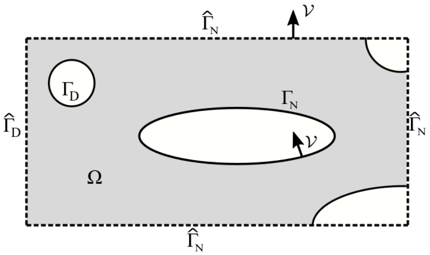
Consider the following linear elasticity problem:

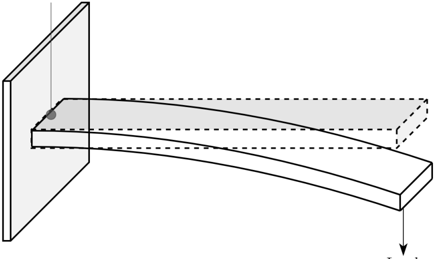
- $\Omega \subset \mathbb{R}^d$
- $\partial\Omega := \Gamma \cup \hat{\Gamma}$
- $\Gamma_D \cap \Gamma_N, \hat{\Gamma}_D \cap \hat{\Gamma}_N = \emptyset$
- u : displacement field
- μ, λ are Lamé parameters
- $\epsilon(\nabla u) := [\nabla u + \nabla u^T]/2$

$$\begin{cases} -Div(\sigma) = f & \text{in } \Omega \\ \sigma = 2\mu\epsilon(\nabla u) + \lambda tr(\epsilon(\nabla u))I & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \cup \hat{\Gamma}_D \\ \sigma \cdot \nu = g_N & \text{on } \Gamma_N \cup \hat{\Gamma}_N \end{cases} \tag{1}$$

Where $\hat{\Gamma}$ is the inactive boundary and Γ is the active boundary.

Domain





- The design domain $\hat{\mathcal{D}}$ is the dashed rectangular region. The fixed background mesh.
- Γ is the active boundary and is described by a levelset function.
- $\hat{\Gamma}$ is the inactive boundary and $\hat{\Gamma} = \partial\hat{\mathcal{D}} \cap \partial\Omega$

We view (1) as solving for the displacement of a cantilever beam.

Weak Formulation

We define the following linear and bilinear forms:

$$\chi(\Omega; v) := (f, v)_\Omega + (g_N, v)_{\Gamma_N \cup \hat{\Gamma}_N}$$
$$a(\Omega; u, v) := 2\mu(\epsilon(\nabla u), \epsilon(\nabla v))_\Omega + \lambda(\nabla \cdot u, \nabla \cdot v)_\Omega$$

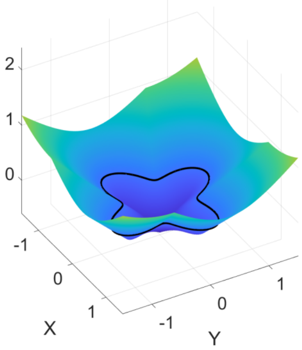
And we state the weak formulation of (1):

Let $V_D := \{v \in H^1(\Omega) : v|_{\Gamma_D \cup \hat{\Gamma}_D} = 0\}$ and find $u \in V_D$ such that

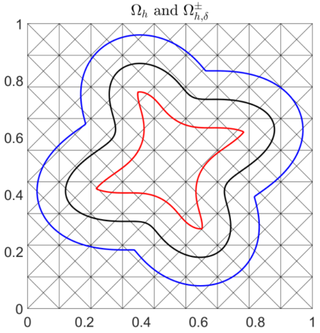
$$a(\Omega; u, v) = \chi(\Omega; v) \quad \forall v \in V_D(\Omega) \tag{2}$$

Levelset Formulation

- $\phi_h : \mathbb{R}^d \to \mathbb{R}$ a Lipschitz levelset function



- $\Omega_h := \{x \in \hat{\mathcal{D}} : \phi_h(x) < 0\}$
- $\Gamma_h := \{x \in \hat{\mathcal{D}} : \phi_h(x) = 0\}$



Stabilization Forms

We define the stabilized Nitsche Form:

$$
\begin{aligned}
a_h(\Omega_h; u, v) :=& a(\Omega_h; u, v) - (\sigma(u)\nu, v)_{\Gamma_{h,D}} - (u, \sigma(v)\nu)_{\Gamma_{h,D}} \\
& \gamma_D h^{-1} b(\Omega_h; u, v) + \gamma_N h(\sigma(u)\nu, \sigma(v)\nu)_{\Gamma_{h,N}} \\
b(\Omega_h; u, v) :=& 2\mu(u, v)_{\Gamma_{h,D}} + \lambda(u \cdot \nu, v \cdot \nu)_{\Gamma_{h,D}} \\
\chi_h(\Omega_h; u, v) :=& \chi(\Omega_h; v) + \gamma_N h(g_N, \sigma(v)\nu)_{\Gamma_{h,N}}
\end{aligned}
$$

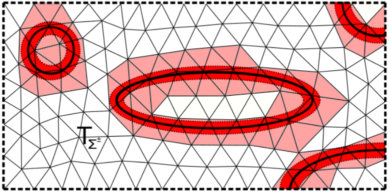
- where $\gamma_D > 0$ and $\gamma_N \geq 0$ (we choose $\gamma_N = 0$)

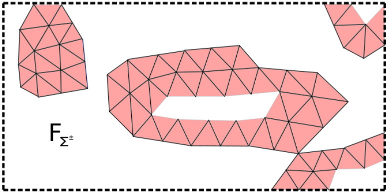
And we define the Facet stabilization form:

$$
s_{h,F}(u, v) := h^{-2} \int_{\omega_F} (u_1 - u_2) \cdot (v_1 - v_2) dx
$$

- T_1 and T_2 neighboring elements
- $\omega_F := T_1 \cup T_2$
- u_i, v_i is the canonical extension of $u_i|_{T_i}$ and $v_i|_{T_i}$
- $F := T_1 \cap T_2$

Unfitted Finite Element Scheme





We define a new bilinear form including the stabilization forms:

$$A_h(\Omega; u, v) := a_h(\Omega; u, v) + \gamma_s s_h\left(\mathcal{F}_{\Sigma^\pm_{\delta,D}}; u, v\right) + \gamma_s h^2 s_h\left(\mathcal{F}_{\Sigma^\pm_{\delta,N}}; u, v\right)$$

Full Scheme:
Find $u_h \in V_h(\Omega_h)$ (discrete finite element space) such that

$$A_h(\Omega_h; u_h, v_h) = \chi_h(\Omega_h; v_h), \quad \forall v_h \in V_h(\Omega_h) \tag{3}$$

Minimization Problem

We focus on minimizing the compliance. Compliance measures the internal elastic energy and is defined as

$$J(\Omega; v) := \chi(\Omega; v) + a_0|\Omega|, \qquad a_0 > 0 \tag{4}$$

We state our minimization problem:

$$J(\Omega_{h,min}; u_h(\Omega_{h,min})) = \min_{\forall \Omega_h \in \mathcal{A},\ \forall v_h \text{ solving (3)}} J(\Omega_h; v_h) \tag{5}$$

- $\mathcal{A}$ is the set of admissible shapes
- Ω_h is the discrete domain

Lagrangian Formulation

In order to free the PDE constraint of our minimization problem (5), we introduce a Lagrangian

$$\begin{aligned} L(\Omega_h; v_h, q_h) :=& J(\Omega_h; v_h) - A_h(\Omega_h; v_h, q_h) + \chi_h(\Omega_h; q_h) \\ L(\overline{\Omega}_h; \overline{u}_h, \overline{p}_h) =& \min_{\forall \Omega_h \in \mathcal{A}, \forall v_h \in V_h(\Omega_h)} \max_{\forall q_h \in V_h(\Omega_h)} L(\Omega_h; v_h, q_h) \end{aligned} \tag{6}$$

Which leads to the first order conditions:

$$\begin{aligned} \delta_{q_h} L(\overline{\Omega}_h; \overline{u}_h, \overline{p}_h)(v_h) =& 0 \qquad\qquad \delta_{\Omega_h} L(\overline{\Omega}_h; \overline{u}_h, \overline{p}_h)(Y) = 0 \\ \delta_{v_h} L(\overline{\Omega}_h; \overline{u}_h, \overline{p}_h)(w_h) =& 0 \end{aligned}$$

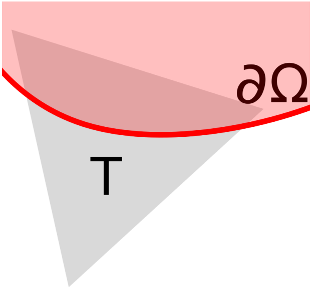
which implies that $\overline{u}_h$ and $\overline{p}_h$ solve the variational problems

$$\begin{aligned} A_h(\overline{\Omega}_h; \overline{u}_h, v_h) =& \chi_h(\overline{\Omega}_h; v_h) \qquad && \forall v_h \in V_h(\overline{\Omega}_h) \\ A_h(\overline{\Omega}_h; w_h, \overline{p}_h) =& \delta_{v_h} J(\overline{\Omega}_h; v_h)(w_h) \qquad && \forall w_h \in V_h(\overline{\Omega}_h) \end{aligned}$$

Shape Optimization Scheme

Let $f(x)$ be defined on $\hat{\mathcal{D}}$ and independent of the shape Ω and recall the shape derivative of a bulk shape functional

$$J(\Omega) := \int_{\Omega} f(x) dx \qquad \delta_{\Omega} J(\Omega)(V) = \int_{\partial\Omega} f(a) V(a,0) \cdot \nu \, dS(a)$$



We assume that no segment of $\partial\Omega$ lies directly on any facet. With some work, one can obtain the exact shape derivative on a cut element to obtain:

$$J_T(\Omega) := \int_{\Omega \cap T} f(x) dx$$

$$\delta_{\Omega} J_T(\Omega)(V) = \int_{\partial\Omega \cap T} f(a) V(a,0) \cdot \nu \, dS(a)$$

We make the following assumptions

- $\hat{\Gamma} = \hat{\Gamma}_D \cup \hat{\Gamma}_N$ is fixed.
- $\Gamma_D = \emptyset$
- $f = 0$
- $g \neq 0$ on a subset of $\hat{\Gamma}_N$

We can compute the following shape derivatives over discrete functionals

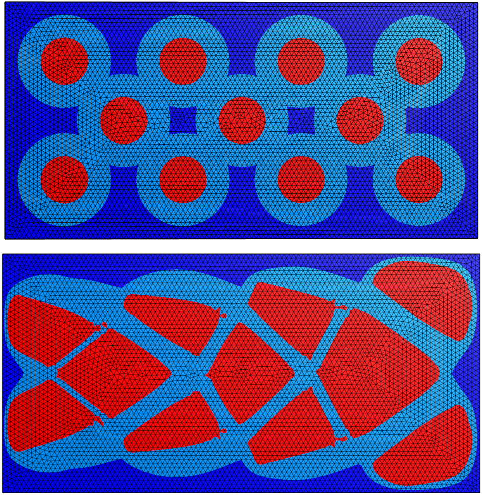
$$\delta_{\Omega_h} A_h(\Omega_h; u_h, v_h)(Y) = \int_{\Gamma_h} 2\mu\epsilon(\nabla u_h) : \epsilon(\nabla v_h) + \lambda(\nabla \cdot u_h)(\nabla \cdot v_h))Y \cdot \nu$$

$$\delta_{\Omega_h} \chi_h(\Omega_h; v_h)(Y) = 0 \qquad \delta_{\Omega_h} J(\Omega_h; v_h)(Y) = a_0 \int_{\Gamma_h} (Y \cdot \nu)$$

$$\delta_{\Omega_h} L(\Omega_h; u_h, u_h)(Y) = \int_{\Gamma_h} 2\mu|\epsilon(\nabla u_h)|^2 + \lambda|\nabla \cdot u_h|^2 + a_0)Y \cdot \nu$$

For all $v_h \in V_h(\Omega_h)$ and all admissible shape permutations Y.

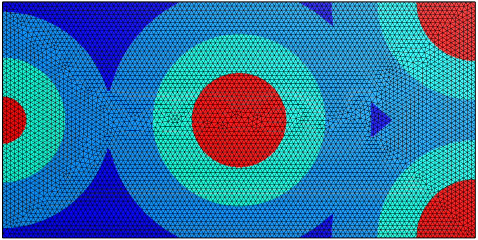
Cantilever with 11 holes, $g_N = (0, -1)$ on far right

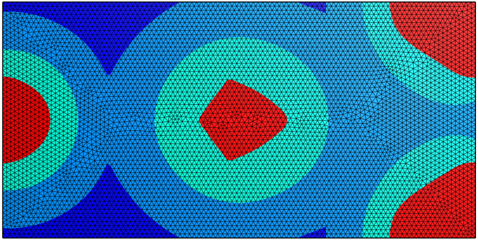


We start with an initial shape (top) having 11 holes spaced evenly and we do not update the levelset function along the boundary so that it retains its rectangular shape.

The resulting image (bottom) is shown

Cantilever with 4 holes, $g_N = (0, -1)$ on far right

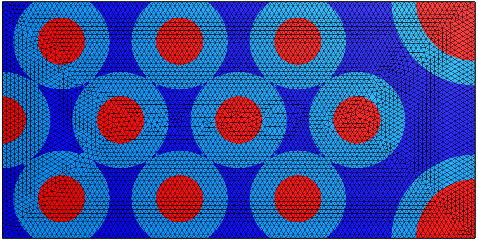


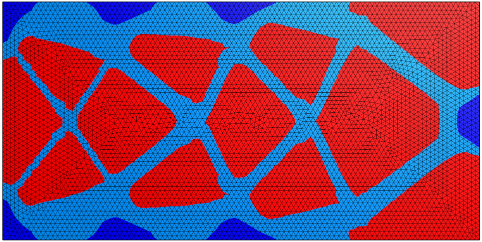


This time we allow the levelset function to be updated on a portion of the boundary, but still restrict the levelset on the boundary depicted in black here:

The initial guess (top) and resulting shape (bottom) are displayed.

Cantilever with 12 holes, $g_N = (0, -1)$ on far right





We again restrict the update of the levelset on the boundary depicted in black here:

And from this simulation we produce qualitative results similar to that in [3]

References

[1] Christopher Basting and Dmitri Kuzmin. "A minimization-based finite element formulation for interface-preserving level set reinitialization". In: *Computing* Vol. 95 (May 2012).

[2] Erik Burman and Peter Hansbo. "Fictitious domain finite element methods using cut elements: II. A stabilized Nitsche method". In: *Applied Numerical Mathematics* 62.4 (2012), pp. 328–341. ISSN: 0168-9274.

[3] Erik Burman et al. "Shape optimization using the cut finite element method". In: *Computer Methods in Applied Mechanics and Engineering* 328 (2018), pp. 242–261. ISSN: 0045-7825.

[4] Shawn W. Walker. *The Shapes of Things: A Practical Guide to Differential Geometry and the Shape Derivative*. 1st. Vol. 28. Advances in Design and Control. SIAM, 2015.