# Shape Optimization with an Unfitted Finite Element Method


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## Abstract

We present a formulation of a PDE-constrained shape optimization problem that uses an unfitted finite element method (FEM). The geometry is represented (and optimized) using a level set approach and we consider objective functionals that are defined over bulk domains. For a discrete objective functional (i.e. one defined in the unfitted FEM framework), we show that the exact shape derivative can be computed rather easily. In other words, one gains the benefits of both the optimize-then-discretize and discretize-then-optimize approaches.

We illustrate the method on a simple model (geometric) problem with known exact solution, as well as shape optimization of structural designs. We also give some discussion on convergence of minimizers.

This is joint work with Shawn W. Walker (walker@math.lsu.edu, LSU)


## Model Problem

Consider the following linear elasticity problem:

- $\Omega \in \mathbb{R}^d$
- $\partial\Omega := \Gamma \cup \hat{\Gamma}$
- $\Gamma_D \cap \Gamma_N, \hat{\Gamma}_D \cap \hat{\Gamma}_N = \emptyset$
- $u$ : displacement field
- $\mu$, $\lambda$ are Lamé parameters
- $\epsilon(\nabla u) := [\nabla u + \nabla u^T]/2$

$$\begin{aligned} -Div(\sigma) &= f && \text{in } \Omega \\ \sigma &= 2\mu\epsilon(u) + \lambda tr(\epsilon(u))I && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma_D \cup \hat{\Gamma}_D \\ \sigma \cdot \nu &= g_N && \text{on } \Gamma_N \cup \hat{\Gamma}_N \end{aligned}$$

With "hold-all" domain $\hat{\mathcal{D}}$ and $\Omega$ given as:

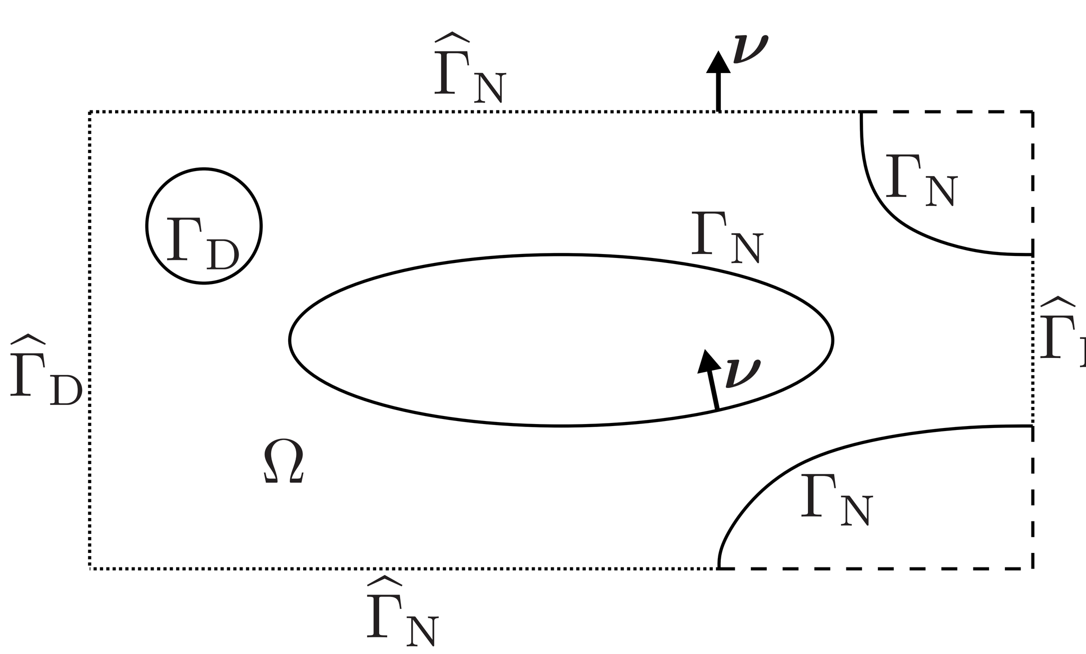



Figure 1:

We define the following linear and bilinear forms:

$$\chi(\Omega, v) := (f, v)_\Omega + (g_N, v)_{\Gamma_N \cup \hat{\Gamma}_N}$$
$$a(\Omega; u, v) := 2\mu(\epsilon(\nabla u), \epsilon(\nabla v))_\Omega + \lambda(\nabla \cdot u, \nabla \cdot v)_\Omega$$

### Weak Formulation

Let $V_D := \{v \in H^1(\Omega) : v|_{\Gamma_D \cup \hat{\Gamma}_D} = 0\}$ and find $u \in V_D$ such that

$$a(\Omega, u, v) = \chi(\Omega, v) \quad \forall v \in V_D(\Omega) \tag{1}$$

We will use the notation $u(\Omega)$ to emphasize the dependence of the unique solution on $\Omega$

We minimize the compliance shape functional given by

$$J(\Omega, v) := \chi(\Omega, v) + a_0|\Omega|, \qquad a_0 > 0$$

Compliance measures the elastic energy. A shape with low compliance will be stiff.

## Levelset Formulation

- $\Gamma$ active boundary
- $\hat{\Gamma} \subset \hat{\mathcal{D}}$ inactive boundary

$$\begin{aligned} \Omega &:= \{x \in \hat{\mathcal{D}} : \phi(x) < 0\} \\ \Gamma &:= \{x \in \hat{\mathcal{D}} : \phi(x) = 0\} \\ \hat{\Gamma} &:= \partial\hat{\mathcal{D}} \cap \partial\Omega \end{aligned}$$

### Perturbed Levelset Function

$$\hat{\phi}(x, t) = \phi(x) + t\eta(x) \tag{2}$$

- $\Omega_t := \{x \in \hat{\mathcal{D}} : \hat{\phi}(x, t) < 0\}$
- $\Gamma_t := \{x \in \hat{\mathcal{D}} : \hat{\phi}(x, t) = 0\}$

Consider the ODE describing the trajectory $x(t)$

$$\dot{x} = V \qquad \forall t > 0$$
$$x(0) = a \in \hat{\mathcal{D}}$$

If $a \in \Gamma_0$ then $\hat{\phi}(x(t), t) = 0$ for all sufficiently small $t$ provided that

$$V(x, t) = -\frac{\nabla\hat{\phi}(x, t)}{|\nabla\hat{\phi}(x, t)|^2}\eta(x)$$

Additionally, we restrict $\eta(x)$ to 0 on $\Sigma \subset \partial\hat{\mathcal{D}}$ in order to force the shape to remain feasible.

Figure 2: Example choice of $\Sigma$ (black)

## Stabilization Forms

### Stabilized Nitsche Form

$$\begin{aligned} a_h(\Omega_h; u, v) :=& a(\Omega_h; u, v) - (\sigma(u)\nu, v)_{\Gamma_{h,D}} - (u, \sigma(v)\nu)_{\Gamma_{h,D}} \\ & \gamma_D h^{-1} b(\Omega_h; u, v) + \gamma_N h(\sigma(u)\nu, \sigma(v)\nu)_{\Gamma_{h,N}} \\ b(\Omega_h; u, v) :=& 2\mu(u, v)_{\Gamma_{h,D}} + \lambda(u \cdot \nu, v \cdot \nu)_{\Gamma_{h,D}} \\ \chi_h(\Omega_h; u, v) :=& \chi(\Omega_h; v) + \gamma_N h(g_N, \sigma(v)\nu)_{\Gamma_{h,N}} \end{aligned}$$

- where $\gamma_D > 0$ and $\gamma_N \geq 0$ (we choose $\gamma_N = 0$)

### Local Stabilization Form

$$s_{h,F}(u, v) := h^{-2}\int_{\omega_F}(u_1 - u_2) \cdot (v_1 - v_2)dx$$

- $T_1$ and $T_2$ neighboring elements
- $\omega_F := T_1 \cup T_2$
- $u_i, v_i$ is the canonical extension of $u_i|_{T_i}$ and $v_i|_{T_i}$
- $F := T_1 \cap T_2$

The global stabilization form for a set of facets $\mathcal{F}$ is given by:

$$s_h(\mathcal{F}; u, v) := \sum_{F \in \mathcal{F}} s_{h,F}(u, v)$$

### Full Scheme

$$A_h(\Omega; u, v) := a_h(\Omega; u, v) + \gamma_s s_h\left(\mathcal{F}_{\Sigma^{\pm}_{\delta,D}}; u, v\right) + \gamma_s h^2 s_h\left(\mathcal{F}_{\Sigma^{\pm}_{\delta,N}}; u, v\right)$$

Find $u_h \in V_h(\Omega_h)$ (the finite element space) such that

$$A_h(\Omega_h; u_h, v_h) = \chi_h(\Omega_h; v_h), \quad \forall v_h \in V_h(\Omega_h) \tag{3}$$

### Minimization Problem

$$J(\Omega_{h,min}, u_h(\Omega_{h,min})) = \min_{\forall\Omega_h \in \mathcal{A},\ \forall v_h \text{ solving } (3)} J(\Omega_h; v_h) \tag{4}$$

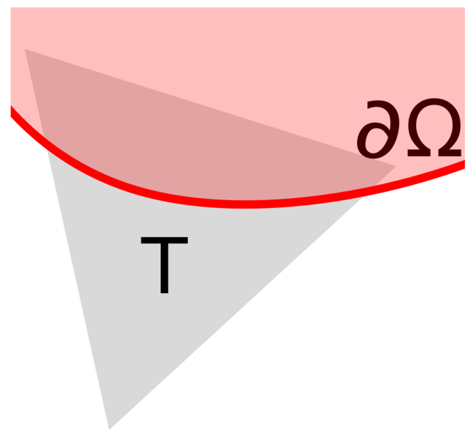
- $\mathcal{A}$ is the set of admissible shapes
- $\Omega_h$ is the discrete domain

## Shape Optimization Scheme

### Shape Derivative

$$J(\Omega) := \int_\Omega f(x)dx \qquad \delta_\Omega J(\Omega)(V) = \int_{\partial\Omega} f(a)V(a, 0) \cdot \nu \, dS(a)$$

Given $f(x)$ defined on $\hat{\mathcal{D}}$ and independent of the shape $\Omega$




With some work one can obtain the exact shape derivative on a cut element

Figure 3: Cut Element

### Shape Derivative on a Cut Element

$$J_T(\Omega) := \int_{\Omega\cap T} f(x)dx \qquad \delta_\Omega J_T(\Omega)(V) = \int_{\partial\Omega\cap T} f(a)V(a, 0) \cdot \nu \, dS(a) \tag{5}$$

Given $f(x)$ defined on $\hat{\mathcal{D}}$ and independent of the shape $\Omega$

### Lagrange Formulation

$$L(\Omega_h; v_h, q_h) := J(\Omega_h; v_h) - A_h(\Omega_h; v_h, q_h) + \chi_h(\Omega_h; q_h)$$
$$L(\overline{\Omega}_h; \overline{u}_h, \overline{p}_h) = \min_{\forall\Omega_h \in \mathcal{A}, \forall v_h \in V_h(\Omega_h)} \max_{\forall q_h \in V_h(\Omega_h)} L(\Omega_h; v_h, q_h) \tag{6}$$

### First Order Conditions

$$\delta_{q_h} L(\overline{\Omega}_h; \overline{u}_h, \overline{p}_h)(z_h) = 0 \qquad \delta_{\Omega_h} L(\overline{\Omega}_h; \overline{u}_h, \overline{p}_h)(Y) = 0$$
$$\delta_{v_h} L(\overline{\Omega}_h; \overline{u}_h, \overline{p}_h)(w_h) = 0$$

which implies that $\overline{u_h}$ and $\overline{p_h}$ solve the variational problems

$$\begin{aligned} A_h(\overline{\Omega}_h; \overline{u}_h, v_h) &= \chi_h(\overline{\Omega}; v_h) && \forall v_h \in V_h(\overline{\Omega}_h) \\ A_h(\overline{\Omega}_h; w_h, \overline{p}_h) &= \delta_{v_h} J(\overline{\Omega}_h; v_h)(w_h) && \forall w_h \in V_h(\overline{\Omega}_h) \end{aligned}$$

We make some additional assumptions:

- $\hat{\Gamma} = \hat{\Gamma}_D \cup \hat{\Gamma}_N$ is fixed.
- $\Gamma_D = \emptyset$
- $f = 0$
- $g \neq 0$ on a subset of $\hat{\Gamma}_N$

Due to these assumptions and (5) we have the following exact shape derivatives:

$$\delta_{\Omega_h}\chi_h(\Omega_h; v_h)(Y) = 0$$
$$\delta_{\Omega_h} J(\Omega_h; v_h)(Y) = a_0 \int_{\Gamma_h}(Y \cdot \nu)$$
$$\delta_{\Omega_h} A_h(\Omega_h; u_h, v_h)(Y) = \int_{\Gamma_h} 2\mu\epsilon(\nabla u_h) : \epsilon(\nabla v_h) + \lambda(\nabla \cdot u_h)(\nabla \cdot v_h))Y \cdot \nu$$

For all $v_h \in V_h(\Omega_h)$ and all admissible shape permutations $Y$.

$$\delta_{\Omega_h} L(\Omega_h; u_h, u_h)(Y) = \int_{\Gamma_h} 2\mu|\epsilon(\nabla u_h)|^2 + \lambda|\nabla \cdot u_h|^2 + a_0)Y \cdot \nu$$

## Future Directions

- Make sure the perturbed domain is still feasible
- Boundary functionals are an issue

## Simulations

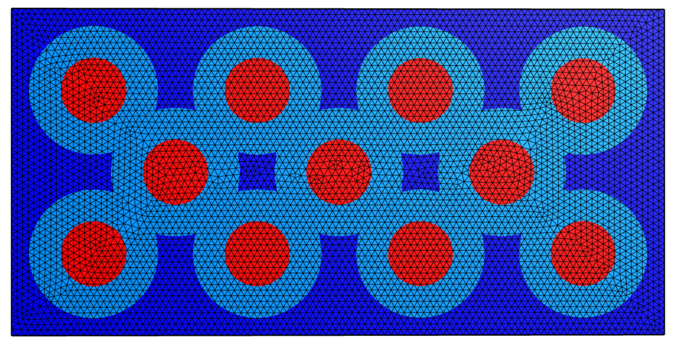


Figure 4: Initial Guess

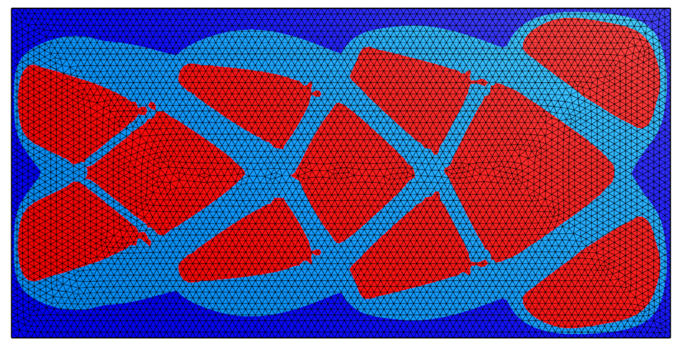


Figure 5: Resulting Domain

The levelset remains unchanged along the boundary $\partial\hat{\mathcal{D}}$

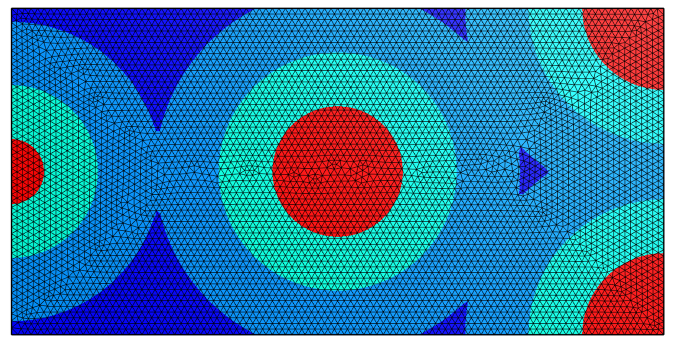


Figure 6: Initial Guess

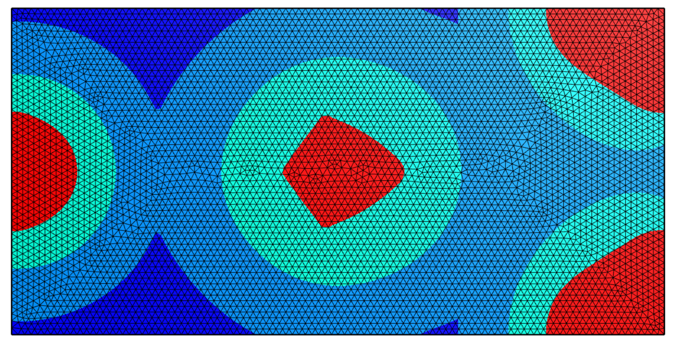


Figure 7: Resulting Domain

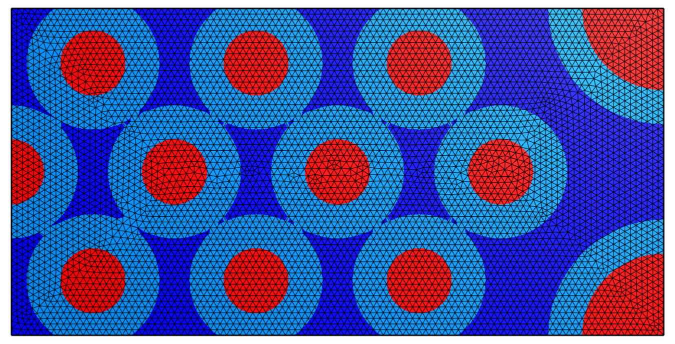


Figure 8: Initial Guess

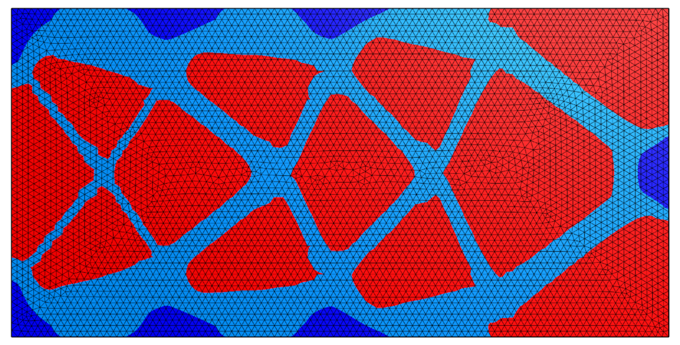


Figure 9: Resulting Domain

We chose the initial domain in to try to replicate the qualitative results of [3] and we produce a shape nearly identical. The levelset remains unchanged on $\Sigma$ as in Figure 2.

## References


[1] Christopher Basting and Dmitri Kuzmin. "A minimization-based finite element formulation for interface-preserving level set reinitialization". In: *Computing* Vol. 95 (May 2012).

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[4] Shawn W. Walker. *The Shapes of Things: A Practical Guide to Differential Geometry and the Shape Derivative*. 1st. Vol. 28. Advances in Design and Control. SIAM, 2015.